

Weighted discrete hypergroups

MAHMOOD ALAGHMANDAN AND EBRAHIM SAMEI

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Abstract

Weighted group algebras have been studied extensively in Abstract Harmonic Analysis where complete characterizations have been found for some important properties of weighted group algebras, namely amenability and Arens regularity. One of the generalizations of weighted group algebras is weighted hypergroup algebras. Defining weighted hypergroups, analogous to weighted groups, we study Arens regularity and isomorphism to operator algebras for them. We also examine our results on three classes of discrete weighted hypergroups constructed by conjugacy classes of FC groups, the dual space of compact groups, and hypergroup structure defined by orthogonal polynomials. We observe some unexpected examples regarding Arens regularity and operator isomorphisms of weighted hypergroup algebras.

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1 Introduction

Discrete hypergroups were defined as a generalization of (discrete) groups. Also, some objects related to locally compact groups may be studied as discrete hypergroups. For instance, double cosets of a locally compact group with respect to a compact open subgroup. In particular, this class includes the hypergroup structures on conjugacy classes of an FC group (i.e. every conjugacy class is finite). Also, for a compact group G , the set of equivalence classes of irreducible (unitary) representations of G , denoted by \widehat{G} and called the *dual of the group G* , is a commutative discrete hypergroup. On one hand these examples as well as hypergroups defined by orthogonal polynomials connect the studies done on hypergroups to different topics in abstract harmonic analysis. On the other hand, the similarities of hypergroups and groups suggest that one may be able to generalize the studies on groups to hypergroups.

One of the topics related to hypergroups which has been initiated based on a similar study on groups is *weighted hypergroups* and *weighted hypergroup algebras*, as they are defined in the following. The weighted hypergroup algebra, as a Banach algebra can be the subject of study for different properties of Banach algebras. The first studies over weighted hypergroup algebras may be tracked back to [4, 11, 12].

In this manuscript, we study Arens regularity and isomorphism to operator algebras for weighted hypergroup algebras. To recall, the second dual of a Banach algebra can be equipped with two algebraic actions to form Banach algebras, we call a Banach algebra ‘Arens regular’ if these two actions coincide. Also a Banach algebra \mathcal{A} is called an *operator algebra* if there is a Hilbert space \mathcal{H} such that \mathcal{A} is a closed subalgebra of $\mathcal{B}(\mathcal{H})$. The main result of [23] rules out Arens regularity (and subsequently operator algebra isomorphism) of weighted hypergroup algebras for non-discrete hypergroups. Consequently, this paper is only dedicated to discrete hypergroups, although many results proved in Sections 3 hold for weights on non-discrete hypergroups as well. In this manuscript, we particularly examine our results on various classes of weighted hypergroup algebras with respect to these properties. One may note that, for the specific weight $\omega \equiv 1$, the weighted case is reduced back to regular hypergroups and their algebras.

The paper is organized as follows. We start this paper by Section 2 wherein we give the definition of discrete hypergroups consistently and briefly go through three classes of hypergroup structures we use in examples. Section 3 is devoted to weights on (discrete) hypergroups, their corresponding algebras, and their examples. We continue this section by studying some examples. In particular, in Subsection 3.3, we introduce and study some hypergroup weights on the dual of compact groups.

Arens regularity of weighted group algebras has been studied by Craw and Young in [8]. They showed that a locally compact group G has a weight ω such that $L^1(G, \omega)$ is Arens regular if and only if G is discrete and countable. They also characterized the Arens regularity of weighted group algebras with respect to one feature of the (group) weight, called *0-clusterness* as described in [9]. In Section 4, the Arens regularity of weighted hypergroup algebras for discrete hypergroups is studied and it is shown (Theorem 4.4) that the strong 0-clusterness of the corresponding hypergroup weight results in the Arens regularity of the weighted hypergroup algebra (strong 0-clusterness implies 0-clusterness, [9]).

Injectivity and equivalently isomorphism of weighted group algebras to operator algebras have been studied before, see [19, 24]. In Section 5, studying the hypergroup case, we demonstrate (Theorem 5.2) that for hypergroup weights which are weakly additive and whose inverse is 2-summable over the hypergroup, the weighted hypergroup algebra is injective and hence an isomorphism to an operator algebra exists. To do so, we apply some results regarding Littlewood multipliers of hypergroups. This machinery lets us to examine a class of hypergroup weights which are not weakly additive, namely exponential weights, in Subsection 5.1.

In Sections 4 and 5 we present many examples to highlight some unexpected contrasts with some results in the theory of weighted Fourier algebras on compact groups (Examples 4.6, 4.10 and 5.5).

Some results of this paper were first presented in the first author’s Ph.D. thesis, [2], under the supervision of Yemon Choi and Ebrahim Samei.

2 Discrete hypergroups and examples

In this paper, H is always a discrete hypergroup in the sense of [15] unless otherwise is stated. For basic definitions and facts we refer the reader to the fundamental paper of Jewett, [15], or the

comprehensive book [6].

2.1 Definition

Let H be a discrete set. Let $\ell^1(H)$ denote the Banach space of all functions (bounded measures) $f : H \rightarrow \mathbb{C}$ which are absolutely summable with respect to the counting measure, i.e. $\|f\|_1 := \sum_{x \in H} |f(x)| < \infty$. Let $c_c(H)$ and $c_0(H)$ denote respectively the space of all finitely supported and vanishing at infinity elements of $\ell^\infty(H)$. We call H a *discrete hypergroup* if the following conditions hold.

- (H1) There exists an associative binary operation $*$ called *convolution* on $\ell^1(H)$ under which $\ell^1(H)$ is a Banach algebra. Moreover, for every x, y in H , $\delta_x * \delta_y$ is a positive measure with a finite support and $\|\delta_x * \delta_y\|_{\ell^1(H)} = 1$.
- (H2) There exists an element (necessarily unique) e in H such that $\delta_e * \delta_x = \delta_x * \delta_e = \delta_x$ for all x in H .
- (H3) There exists a (necessarily unique) bijection $x \rightarrow \check{x}$ of H called *involution* satisfying $(\delta_x * \delta_y)^\check{\check{}} = \delta_{\check{y}} * \delta_{\check{x}}$ for all $x, y \in H$.
- (H4) e belongs to $\text{supp}(\delta_x * \delta_y)$ if and only if $y = \check{x}$.

We call a hypergroup H *commutative* if $\ell^1(H)$ forms a commutative algebra. The *left translation* on $\ell^\infty(H)$ is defined by $L_x f : H \rightarrow \mathbb{C}$ where $L_x f(y) := f(\delta_{\check{x}} * \delta_y)$ for each f in $\ell^\infty(H)$ and $x, y \in H$. A non-zero, positive, left invariant linear functional h (possibly unbounded) on $c_c(H)$ is called a *Haar measure*, [15]. For a discrete hypergroup the existence of a Haar measure is proved and it is unique up to multiplication by a positive constant. Indeed, for a discrete hypergroup, a Haar measure $h : H \rightarrow (0, \infty)$ such that $h(e) = 1$ is defined by $h(x) = (\delta_{\check{x}} * \delta_x(e))^{-1}$ for all $x \in H$.

The *hypergroup algebra*, denoted by $L^1(H, h)$ is the Banach algebra of integrable functions on H with respect to the Haar measure h equipped with the convolution $f *_h g := \sum_{x \in H} f(x) L_x g h(x)$. It is easy to observe that $f \mapsto fh$ is an isometric algebra isomorphism from the Banach algebra $L^1(H, h)$ onto the Banach algebra $\ell^1(H)$. Due to this isomorphism, we focus our study on $\ell^1(H)$ without loss of generality.

2.2 The conjugacy classes of FC groups

Let G be a (discrete) group with the group algebra $\ell^1(G)$ and $\text{Conj}(G)$ is the set of all conjugacy classes of G . We denote the centre of the group algebra by $Z\ell^1(G)$. The group G is called an *FC* or *finite conjugacy group* if for each $C \in \text{Conj}(G)$, $|C| < \infty$. For such groups, $\text{Conj}(G)$ forms a commutative discrete hypergroup (which is the discrete case of the hypergroup structures defined in [15, Subsection 8.3]). Let Ψ denote the linear mapping from $Z\ell^1(G)$ to $\ell^1(\text{Conj}(G))$ defined by $\Psi(f)(C) = |C|f(C)$ for $C \in \text{Conj}(G)$ where $\Psi(f)(C) := f(x)$ for (every) $x \in C$. Then one can easily check that Ψ is an isometric Banach algebra isomorphism between $\ell^1(\text{Conj}(G))$ and $Z\ell^1(G)$.

As an extension of finite products of hypergroups (or in particular groups), let $\{H_i\}_{i \in \mathbf{I}}$ be a family of discrete hypergroups, then $H := \bigoplus_{i \in \mathbf{I}} H_i$ where for each $x \in H$, $x = (x_i)_{i \in \mathbf{I}}$ where x_i is the identity of the hypergroup H_i , e_{H_i} , for all $i \in \mathbf{I}$ except finitely many. H is called *restricted direct product* of $\{H_i\}_{i \in \mathbf{I}}$ which is a hypergroup (or a group if for every i , H_i is a group).

Example 2.1 For a family of FC groups $\{G_i\}_{i \in \mathbf{I}}$, let $G := \bigoplus_{i \in \mathbf{I}} G_i$ be the restricted direct product of $\{G_i\}_{i \in \mathbf{I}}$. Then G is a discrete FC group and $\text{Conj}(G)$ is the hypergroup generated by the restricted direct product of $\{\text{Conj}(G_i)\}_{i \in \mathbf{I}}$, $\text{Conj}(G) = \bigoplus_{i \in \mathbf{I}} \text{Conj}(G_i)$.

2.3 The dual of compact groups

Let G be a compact group and \widehat{G} denotes the set of all irreducible unitary (necessary finite-dimensional) representations of a compact group G , up to unitary equivalence relation. It is known that the irreducible decomposition of the tensor products of elements of \widehat{G} leads to a discrete commutative hypergroup structure on \widehat{G} where $h(\pi) = d_\pi^2$ for d_π the dimension of $\pi \in \widehat{G}$. (See [6, Example 1.1.14]).

Example 2.2 Let $\text{SU}(2)$ be the compact Lie group of 2×2 special unitary matrices on \mathbb{C} , and let $\widehat{\text{SU}}(2)$ be the hypergroup of all irreducible representations on $\text{SU}(2)$. It is known that $\widehat{\text{SU}}(2) = (\pi_\ell)_{\ell \in \mathbb{N}_0}$ where $\mathbb{N}_0 := \{0, 1, 2, \dots\}$ and the dimension of π_ℓ is $\ell + 1$. Moreover, for all ℓ, ℓ' , $\overline{\pi}_\ell = \pi_\ell$ and $\pi_\ell \otimes \pi_{\ell'} \cong \pi_{|\ell - \ell'|} \oplus \pi_{|\ell - \ell'| + 2} \oplus \dots \oplus \pi_{\ell + \ell'}$. This tensor decomposition is called “*Clebsch-Gordan*” decomposition formula. So using the Clebsch-Gordan formula, we have that

$$\delta_{\pi_\ell} * \delta_{\pi_{\ell'}} = \sum_{r=|\ell - \ell'|}^{\ell + \ell'} \frac{(r + 1)}{(\ell + 1)(\ell' + 1)} \delta_{\pi_r}$$

where

$$\sum_{r=a}^b f(r) = f(a) + f(a + 2) + \dots + f(b - 2) + f(b). \quad (2.1)$$

Also $\overline{\pi}_\ell = \pi_\ell$ and $h(\pi_\ell) = (\ell + 1)^2$ for all ℓ .

Example 2.3 Suppose that $\{G_i\}_{i \in \mathbf{I}}$ is a non-empty family of compact groups for arbitrary indexing set \mathbf{I} . Let $G := \prod_{i \in \mathbf{I}} G_i$ be the product of $\{G_i\}_{i \in \mathbf{I}}$ i.e. $G := \{(x_i)_{i \in \mathbf{I}} : x_i \in G_i\}$ equipped with the product topology. Then \widehat{G} is the restricted direct product of hypergroups $\{\widehat{G}_i\}_{i \in \mathbf{I}}$, (see [14, Theorem 27.43]).

2.4 Polynomial hypergroups

Let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Hypergroups related to systems of orthogonal polynomials in one variable have been introduced and studied by Lasser [17] and Voit [25]. Such a hypergroup structure on \mathbb{N}_0 is called a *polynomial hypergroup* which is also discrete and commutative ([6, Section 3.2]).

Example 2.4 Let \mathbb{N}_0 be equipped with the hypergroup convolution $\delta_n * \delta_m := (1/2)\delta_{|n-m|} + (1/2)\delta_{n+m}$. This hypergroup structure is called *Chebyshev polynomial of the first type*. One can show that the hypergroup algebra of \mathbb{N}_0 is isomorphic to the subalgebra of symmetric functions on the Fourier algebra of the torus, i.e. $Z_{\pm 1}A(\mathbb{T}) := \{f + \check{f} : f \in A(\mathbb{T})\}$ where $\check{f}(x) = f(-x)$.

3 Weighted discrete hypergroups and examples

In this section we study weights on discrete hypergroups, their corresponding algebras, and their examples. Specially we are interested to see concrete examples of weights defined on the classes of commutative discrete hypergroups which were mentioned in Section 2.

3.1 General Theory

We believe all the definitions and observations in this subsection still hold for non-discrete hypergroups (see [11]), but here we are mainly interested in the discrete case.

Definition 3.1 Let H be a discrete hypergroup. We call a function $\omega : H \rightarrow (0, \infty)$ a *weight* if, for every $x, y \in H$, $\omega(\delta_x * \delta_y) \leq \omega(x)\omega(y)$. Then we call (H, ω) a *weighted hypergroup*. Let $\ell^1(H, \omega)$ be the set of all complex functions on H such that

$$\|f\|_{\ell^1(H, \omega)} := \sum_{t \in H} |f(t)|\omega(t) < \infty.$$

Then one can easily observe that $(\ell^1(H, \omega), \|\cdot\|_{\ell^1(H, \omega)})$ equipped with the (extended) convolution of $\ell^1(H)$ forms a Banach algebra which is called a *weighted hypergroup algebra*.

- It is easy to see that if ω is a positive function on H such that $\omega(t) \leq \omega(x)\omega(y)$ for all $t, x, y \in H$ where $t \in \text{supp}(\delta_x * \delta_y)$, then ω is a weight on H . We call such a weight a *central weight*. We will show later that not all hypergroup weights are central. (See Examples 3.10 and 3.11)
- A hypergroup weight ω on H is called *weakly additive*, if for some $C > 0$, $\omega(\delta_x * \delta_y) \leq C(\omega(x) + \omega(y))$ for all $x, y \in H$.
- Two weights ω_1 and ω_2 are called *equivalent* if there are constants C_1, C_2 such that $C_1\omega_1 \leq \omega_2 \leq C_2\omega_1$.

Example 3.2 Let $\{H_i\}_{i \in \mathbf{I}}$ be a family of discrete hypergroups with corresponding weights $\{\omega_i\}_{i \in \mathbf{I}}$ such that $\omega_i(e_{H_i}) = 1$ for all $i \in \mathbf{I}$ except finitely many. Then $\omega(x_i)_{i \in \mathbf{I}} := \prod_{i \in \mathbf{I}} \omega_i(x_i)$ forms a hypergroup weight on the restricted direct product of hypergroups $\{H_i\}_{i \in \mathbf{I}}$.

A discrete hypergroup H is called *finitely generated* if for a finite set $F \subseteq H$ with $F = \check{F}$, we have $H = \bigcup_{n \in \mathbb{N}} F^{*n}$ then F is called a *finite symmetric generator* of H . We define

$$\tau_F : H \rightarrow \mathbb{N} \cup \{0\} \tag{3.1}$$

by $\tau_F(x) := \inf\{n \in \mathbb{N} : x \in F^{*n}\}$ for all $x \neq e$ and $\tau_F(e) = 0$. It is straightforward to verify that if F' is another finite symmetric generator of H , then for some constants C_1, C_2 , $C_1\tau_{F'} \leq \tau_F \leq C_2\tau_{F'}$. If there is no risk of confusion, we may just use τ instead of τ_F .

Definition 3.3 For a given $\beta \geq 0$, $\omega_\beta(x) := (1 + \tau(x))^\beta$ is a central weight on H which is called a *Polynomial weight*. Similarly, for given $C > 0$ and $0 \leq \alpha \leq 1$, $\sigma_{\alpha,C}(x) := e^{C\tau(x)^\alpha}$ is a central weight on H which is called an *Exponential weight*.

Proposition 3.4 Let H, H' be two discrete hypergroups and $\phi : H_1 \rightarrow H_2$ be a surjective hypergroup homomorphism. If ω is a weight on H so that for every $x \in H$, $\omega(x) \geq \delta$ for some $\delta > 0$. Then ω' defined by

$$\omega'(y) := \inf\{\omega(x) : x \in H, \phi(x) = y\} \quad (y \in H'),$$

is a weight on H' .

Proof. Proof is immediate if one note that $\phi : c_c(H) \rightarrow c_c(H')$ satisfies $\|\phi(f)\|_{\ell^1(H', \omega')} \leq \|f\|_{\ell^1(H, \omega)}$ and

$$\omega'(\delta_{\phi(x)} * \delta_{\phi(z)}) = \|\delta_{\phi(x)} * \delta_{\phi(z)}\|_{\ell^1(H', \omega')} = \|\phi(\delta_x * \delta_z)\|_{\ell^1(H', \omega')} \leq \|\delta_x\|_{\ell^1(H, \omega)} \|\delta_z\|_{\ell^1(H, \omega)} = \omega(x)\omega(z)$$

for every pair $x, z \in H$. □

3.2 Weights on $\text{Conj}(G)$

Let (G, σ) be a weighted group i.e. $\sigma(xy) \leq \sigma(x)\sigma(y)$ for all $x, y \in G$. We use $\ell^1(G, \sigma)$ to denote the weighted group algebra constructed by σ . Let $Z\ell^1(G, \sigma)$ denote the center of $\ell^1(G, \sigma)$. It is not hard to show that $Z\ell^1(G, \sigma)$ is the set of all $f \in \ell^1(G, \sigma)$ for them $f(yxy^{-1}) = f(x)$ for all $x, y \in G$.

The following proposition lets us apply group weights to generate hypergroup weights on $\text{Conj}(G)$. The proof is straightforward, so we omit it here.

Proposition 3.5 Let G be an FC group possessing a weight σ . Then the mean function ω_σ defined by $\omega_\sigma(C) := |C|^{-1} \sum_{t \in C} \sigma(t)$ ($C \in \text{Conj}(G)$) is a weight on the hypergroup $\text{Conj}(G)$. Further, $\ell^1(\text{Conj}(G), \omega_\sigma)$ is isometrically Banach algebra isomorphic to $Z\ell^1(G, \sigma)$.

Remark 3.6 Let G be an FC group and let ω be a central weight on $\text{Conj}(G)$. Then the mapping σ_ω , defined on G by $\sigma_\omega(x) := \omega(C_x)$, is a group weight on G . And $\ell^1(\text{Conj}(G), \omega)$ as a Banach algebra is isometrically isomorphic to $Z\ell^1(G, \sigma_\omega)$.

Example 3.7 Let G be a discrete FC group. The mapping $\omega(C) = |C|$, for $C \in \text{Conj}(G)$, is a central weight on $\text{Conj}(G)$.

Example 3.8 Let $G = \bigoplus_{i \in \mathbf{I}} G_i$ for a family of finite groups $\{G_i\}_{i \in \mathbf{I}}$. Given $C = (C_i)_{i \in \mathbf{I}} \in \text{Conj}(G)$, define $\mathbf{I}_C := \{i \in \mathbf{I} : C_i \neq e_{G_i}\}$. For each $\alpha > 0$, we define a mapping $\omega_\alpha(C) := (1 + |C_{i_1}| + \dots + |C_{i_n}|)^\alpha$ where $i_j \in \mathbf{I}_C$. We show that ω_α is a central weight on $\text{Conj}(G)$. To do so let $E \subseteq CD$ for some $E, C, D \in \text{Conj}(G)$. One can easily show that for each $i \in \mathbf{I}$, $E_i \subseteq C_i D_i$; $\mathbf{I}_E \subseteq \mathbf{I}_C \cup \mathbf{I}_D$. Therefore,

$$\begin{aligned} \omega_\alpha(C) &= (1 + \sum_{i \in \mathbf{I}_E} |E_i|)^\alpha \leq (1 + \sum_{i \in \mathbf{I}_E} |C_i| |D_i|)^\alpha \quad (\text{by Example 3.7}) \\ &\leq \left(1 + \sum_{i \in \mathbf{I}_C} |C_i|\right)^\alpha \left(1 + \sum_{i \in \mathbf{I}_D} |D_i|\right)^\alpha = \omega_\alpha(C) \omega_\alpha(D). \end{aligned}$$

A group G is called a group with *finite commutator group* or *FD* if its derived subgroup is finite. It is immediate that for a group G , for every $C \in \text{Conj}(G)$, $|C| \leq |G'|$ when G' is the *derived subgroup* of G . Therefore, the order of conjugacy classes of an FD group are uniformly bounded by $|G'|$. The converse is also true, that is for an FC group G , if the order of conjugacy classes are uniformly bounded, then G is an FD group, see [22, Theorem 14.5.11]. The following proposition implies that every hypergroup weight on the conjugacy classes of an FD group which is constructed by a group weight (as given in Proposition 3.5) is equivalent to a central weight. We omit the proof of the following proposition as it is straightforward.

Proposition 3.9 *Let (G, σ) be a weighted FD group. Then the hypergroup weight $\omega_z(C) := |G'|^2 \omega_\sigma(C)$, for $C \in \text{Conj}(G)$, forms a central weight. Here ω_σ is defined as in Proposition 3.5.*

In contrast to Proposition 3.9, we will see in the following examples that there exist weights on FC groups (with infinite derived subgroup) which are not equivalent to any central weight.

Example 3.10 Let S_3 be the symmetric group of order 6. Let ω be defined on $\text{Conj}(S_3)$ by $\omega(C_e) = 1$, $\omega(C_{(12)}) = 2$, and $\omega(C_{(123)}) = 5$. One may verify that ω is a weight on $\text{Conj}(S_3)$. On the other hand, since $5 = \omega(C_{(123)}) \not\leq \omega(C_{(12)})^2 = 4$, ω is not a central weight.

Example 3.11 We generate the restricted direct product $G = \bigoplus_{n \in \mathbb{N}} S_3$. Let us define the weight $\omega' := \prod_{n \in \mathbb{N}} \omega$ on $\text{Conj}(G)$ where ω is the hypergroup weight on $\text{Conj}(S_3)$ defined in Example 3.10. For each $N \in \mathbb{N}$, define $D_N := \prod_{n \in \mathbb{N}} D_n^{(N)} \in \text{Conj}(G)$ where $D_n^{(N)} = C_{(123)}$ for all $n \in 1, \dots, N$ and $D_n^{(N)} = C_e$ otherwise. One can verify that $D_N \in \text{supp}(\delta_{E_N} * \delta_{E_N})$ for $E_N = \prod_{n \in \mathbb{N}} E_n^{(N)} \in \text{Conj}(G)$ with $E_n^{(N)} = C_{(12)}$ for all $n \in 1, \dots, N$ and $E_n^{(N)} = C_e$ otherwise. Therefore

$$\frac{\omega'(D_N)}{\omega'(E_N)^2} = \prod_{n=1}^N \frac{\omega(C_{(123)})}{\omega(C_{(12)})^2} = (5/4)^N \rightarrow \infty$$

where $N \rightarrow \infty$. Hence, ω' is not equivalent to any central weight.

We close this subsection with the following corollary of Lemma 3.4.

Corollary 3.12 *Let G be an FC group, N a normal subgroup of G , and ω a weight on $\text{Conj}(G)$ such that there is some $\delta > 0$ such that $\omega(C) > \delta$, for any $C \in \text{Conj}(G)$. Then the mapping $\tilde{\omega} : \text{Conj}(G/N) \rightarrow \mathbb{R}^+$ defined by $\tilde{\omega}(C_{xN}) := \inf\{\omega(C_{xy}) : y \in N\}$, for $C_{xN} \in \text{Conj}(G/N)$, forms a weight on $\text{Conj}(G/N)$.*

3.3 Weights on duals of compact groups

In this subsection, G is a compact group. We recall that for each $\pi \in \widehat{G}$ and $f \in L^1(G)$,

$$\widehat{f}(\pi) := \int_G f(x) \overline{\pi(x)} dx$$

is the *Fourier transform* of f at π . Let $VN(G)$ denote the group von Neumann algebra of G , i.e. the von Neumann algebra generated by the left regular representation of G . It is well-known that the predual of $VN(G)$, denoted by $A(G)$, is a Banach algebra of continuous functions on G ; it is called the *Fourier algebra* of G . Moreover, for every $f \in A(G)$,

$$\|f\| := \sum_{\pi \in \widehat{G}} d_\pi \|\widehat{f}(\pi)\|_1 < \infty,$$

where $\|\cdot\|_1$ denotes the trace-class operator norm (look at [14, Section 32]).

In an attempt to find the noncommutative analogue of weights on groups, Lee and Samei in [18] defined a *weight on $A(G)$* to be a densely defined (not necessarily bounded) operator W affiliated with $VN(G)$ and satisfying certain properties mentioned in [18, Definition 2.4] (see also [20]). Specially they assume that W has a bounded inverse, W^{-1} , which belongs to $VN(G)$. For a weight W on $A(G)$, the *Beurling-Fourier algebra* denoted by $A(G, W)$ is defined to be the set of all $f \in A(G)$ such that

$$\|f\|_{A(G, W)} := \sum_{\pi \in \widehat{G}} d_\pi \|\widehat{f}(\pi) \circ W\|_1 < \infty.$$

Indeed $(A(G, W), \|\cdot\|_{A(G, W)})$ forms a Banach algebra with pointwise multiplication. For abelian groups, the definition of Beurling-Fourier algebra corresponds the classical weighted group algebra on the dual group. In [18], the authors also studied Arens regularity and isomorphism to operator algebras for Beurling-Fourier algebras.

Definition 3.13 Let G be a compact group and W a weight on $A(G)$. We define a function $\omega_W : \widehat{G} \rightarrow (0, \infty)$ by

$$\omega_W(\pi) := \frac{\|I_\pi \circ W\|_1}{d_\pi} \quad (\pi \in \widehat{G}), \quad (3.2)$$

where $\|\cdot\|_1$ denotes the trace norm and I_π is the identity matrix corresponding to the Hilbert space of π .

As a specific class of weights on the Fourier algebra of a compact group G , in [18] (and independently, in [20]), *central weights* on $A(G)$ are defined. Indeed, [18, Theorem 2.12] implies that each central weight W can be represented by a unique function $\omega_W : \widehat{G} \rightarrow (0, \infty)$ such that $\omega_W(\sigma) \leq \omega_W(\pi_1)\omega_W(\pi_2)$ for all $\pi_1, \pi_2, \sigma \in \widehat{G}$ where $\sigma \in \text{supp}(\delta_{\pi_1} * \delta_{\pi_2})$. In this specific case of operator weights, ω_W matches with our definition in Definition 3.13 for a central weight on the hypergroup \widehat{G} . In the following we show that the same is true for a general weight on $A(G)$ as well.

Let us define $ZA(G, W) := \{f \in A(G, W) : f(yxy^{-1}) = f(x) \text{ for all } x \in G\}$ which is a Banach algebra with pointwise product and $\|\cdot\|_{A(G, W)}$. Note that for the operator weights W where $\omega_W(\pi) = 1$ for every $\pi \in \widehat{G}$, $ZA(G, W) = ZA(G)$. For more on $ZA(G)$, look at [3].

Theorem 3.14 *Let G be a compact group and W a weight on $A(G)$. Then ω_W is a weight on the hypergroup \widehat{G} and the weighted hypergroup algebra $\ell^1(\widehat{G}, \omega_W)$ is isometrically isomorphic to $ZA(G, W)$.*

Proof. Let $\mathcal{X}(G)$ denote the linear span of all the characters of G . First define a linear mapping $\mathcal{T} : \mathcal{X}(G) \rightarrow c_c(\widehat{G})$ by $\mathcal{T}(\chi_\pi) = d_\pi \delta_\pi$ for each $\pi \in \widehat{G}$. Let $f = \sum_{i=1}^n \alpha_i \chi_{\pi_i} \in \mathcal{X}(G)$ for $\pi_i \in \widehat{G}$ and $\alpha_i \in \mathbb{C}$. In this case,

$$\begin{aligned} \|\mathcal{T}(f)\|_{\ell^1(\widehat{G}, \omega)} &= \sum_{i=1}^n |\alpha_i| d_{\pi_i} \omega(\pi_i) = \sum_{i=1}^n |\alpha_i| d_{\pi_i} \frac{\|I_{\pi_i} \circ W\|_1}{d_{\pi_i}} \\ &= \sum_{i=1}^n d_{\pi_i} \left\| \frac{\alpha_i}{d_{\pi_i}} I_{\pi_i} \circ W \right\|_1 = \sum_{i=1}^n d_{\pi_i} \|\alpha_i \widehat{\chi}_{\pi_i}(\pi_i) \circ W\|_1 = \|f\|_{A(G, W)}. \end{aligned}$$

Therefore, \mathcal{T} forms a norm preserving linear mapping. To show that \mathcal{T} is an algebra homomorphism note that $\mathcal{T}(\chi_{\pi_1} \chi_{\pi_2}) = \mathcal{T}(\chi_{\pi_1}) * \mathcal{T}(\chi_{\pi_2})$. It is known that $\mathcal{X}(G)$ is dense in $ZA(G, W)$ and clearly $c_c(\widehat{G})$ is dense in $\ell^1(\widehat{G}, \omega_W)$. So \mathcal{T} can be extended as an algebra isomorphism from $ZA(G, W)$ onto $\ell^1(\widehat{G}, \omega_W)$ which preserves the norm. In particular, $\ell^1(\widehat{G}, \omega_W)$ forms an algebra with respect to its weighted norm and the convolution, and so ω_W is actually a hypergroup weight on \widehat{G} . \square

The proof of the following lemma is straightforward so we omit it here.

Lemma 3.15 *Let G be a compact group and \widehat{G} be the set of all irreducible representations of G as a discrete commutative hypergroup. Then $\omega_\beta(\pi) = d_\pi^\beta = h(d_\pi)^{\beta/2}$ is a central weight for each $\beta \geq 0$.*

In the following, recall \sum_2 defined in (2.1).

Example 3.16 (Lifting weights from \mathbb{Z} to $\widehat{\text{SU}}(2)$) Let σ be a weight on the group \mathbb{Z} . We define

$$\omega_\sigma(\pi_\ell) := \frac{1}{\ell+1} \sum_{r=-\ell}^{\ell} \sigma(r) \quad (\ell \in \mathbb{N}_0). \quad (3.3)$$

Recall that elements of $\widehat{\text{SU}}(2)$ can be regarded as π_ℓ when $\ell \in \mathbb{N}_0$. Suppose that $m, n \in \mathbb{N}_0$ and without loss of generality $n \geq m$. Then,

$$\begin{aligned}
\omega_\sigma(\pi_m)\omega_\sigma(\pi_n) &= \frac{1}{m+1} \sum_{t=-m}^m \sigma(t) \frac{1}{n+1} \sum_{s=-n}^n \sigma(s) \\
&\geq \frac{1}{(m+1)(n+1)} \sum_{t=-m}^m \sum_{s=-n}^n \sigma(t+s) \quad (\dagger) \\
&= \frac{1}{(m+1)(n+1)} \sum_{t=n-m}^{n+m} \sum_{s=-t}^t \sigma(s) \quad (\ddagger) \\
&= \sum_{t=n-m}^{n+m} \frac{(t+1)}{(m+1)(n+1)} \left(\frac{1}{t+1} \sum_{s=-t}^t \sigma(s) \right) \\
&= \sum_{t=n-m}^{n+m} \frac{(t+1)}{(m+1)(n+1)} \omega_\sigma(\pi_t) \\
&= \omega_\sigma(\delta_{\pi_m} * \delta_{\pi_n}).
\end{aligned}$$

To show that the summations (\dagger) and (\ddagger) are equal, let us arrange (\dagger) as follows.

$$\begin{array}{ccccc}
\sigma(-m-n) & +\sigma(-m-n+2) & +\cdots & +\sigma(-m+n-2) & +\sigma(-m+n) \\
+\sigma(-m-n+2) & +\sigma(-m-n+4) & +\cdots & +\sigma(-m+n) & +\sigma(-m+n+2) \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
+\sigma(m-2) & +\sigma(m) & +\cdots & +\sigma(m+n-4) & +\sigma(m+n-2) \\
+\sigma(m) & +\sigma(m+2) & +\cdots & +\sigma(m+n-2) & +\sigma(m+n) .
\end{array}$$

but the sum of all the entries in the first column and the last row is equal to

$$\sum_{s=-m-n}^{m+n} \sigma(s) .$$

The next column and row give

$$\sum_{s=-m-n+2}^{m+n-2} \sigma(s) ,$$

and so on. So by doing this finitely many times, we get (\ddagger) . Indeed, weight ω_σ follows from the recipe of Definition 3.13 using the non-central weight W on $A(\text{SU}(2))$ defined in (A.1) in Appendix A. So instead of the above computations, one also could use Theorem 3.14 to prove that ω_σ is actually a weight on $\widehat{\text{SU}}(2)$.

Fix $\beta > 0$. One may apply the construction in Example 3.16 for

$$\sigma(\ell) := \begin{cases} 1 & 0 \leq \ell \\ (1-\ell)^\beta & \ell < 0 \end{cases} \quad (\ell \in \mathbb{Z}) \quad (3.4)$$

to construct a hypergroup weight ω_σ on $\widehat{\text{SU}}(2)$. Observe that the weight ω_σ is equivalent to the weight ω_β defined in Lemma 3.15. We will see in Section 4, that this particular weight will give interesting classes of examples. To construct weights from subgroups of compact groups, one can look at [20, Proposition 4.11].

3.4 Weights on polynomial hypergroups

Recall that $\widehat{\text{SU}}(2)$ is a particular example of polynomial hypergroup so-called *Chebyshev polynomials*. Similar arguments can be applied to construct hypergroup weights on polynomial hypergroups applying group weights of \mathbb{Z} .

Example 3.17 Let $f : \mathbb{N}_0 \rightarrow \mathbb{R}^+$ be an increasing function such that $f(0) = 1$. Then $\omega_f(n) = f(n) + 2$ is a central weight on \mathbb{N}_0 when it is equipped with the Chebyshev polynomial hypergroup structure of the first type. Applying the argument in Example 2.4, we can see that $\ell^1(\mathbb{N}_0, \omega_f)$ is isomorphic to the symmetric subalgebra of $A(\mathbb{T}, \sigma)$, that is $Z_{\pm 1}A(\mathbb{T}, \sigma) := \{f + \check{f} : f \in A(\mathbb{T}, \sigma)\}$, for the group weight

$$\sigma_f(\ell) := \begin{cases} 1 & \ell \geq 0 \\ f(-\ell) & \ell < 0 \end{cases} \quad (\ell \in \mathbb{Z})$$

4 Arens regularity

In [16, Chaptetr 4], Kamyabi-Gol applied the topological center of hypergroup algebras to prove some results about the hypergroup algebras and their second duals. For example, in [16, Corollary 4.27], he showed that for a (not necessarily discrete and commutative) hypergroup H (which possesses a Haar measure h), $L^1(H, h)$ is Arens regular if and only if H is finite.

Arens regularity of weighted group algebras has been studied by Craw and Young in [8]. They showed that a locally compact group G has a weight ω such that $L^1(G, \omega)$ is Arens regular if and only if G is discrete and countable. The monograph [9] presents a thorough report on Arens regularity of weighted group algebras. In the following we adapt the machinery developed in [9, Section 8] for weighted hypergroups. In [9, Section 3], the authors study repeated limit conditions and give a rich variety of results for them. Here, we will use some of them.

First let us recall the following definitions. Let \mathcal{A} be a Banach algebra. For $f, g \in \mathcal{A}$, $\phi \in \mathcal{A}^*$, and $F, G \in \mathcal{A}^{**}$, we define the following module actions.

$$\begin{aligned} \langle f \cdot \phi, g \rangle &:= \langle \phi, gf \rangle, & \langle \phi \cdot f, g \rangle &:= \langle \phi, fg \rangle \\ \langle \phi \cdot F, f \rangle &:= \langle F, f \cdot \phi \rangle, & \langle F \cdot \phi, f \rangle &:= \langle F, \phi \cdot f \rangle \\ \langle F \diamond G, \phi \rangle &:= \langle G, \phi \cdot F \rangle, & \langle G \square F, \phi \rangle &:= \langle G, F \cdot \phi \rangle. \end{aligned}$$

Let $F, G \in \mathcal{A}^{**}$, and let $(f_\alpha)_\alpha$ and $(g_\beta)_\beta$ be nets in \mathcal{A} such that $f_\alpha \rightarrow F$ and $g_\beta \rightarrow G$ in the weak* topology. One may show that for products \square and \diamond of \mathcal{A}^{**} ,

$$F \square G = w^* - \lim_{\alpha} w^* - \lim_{\beta} f_\alpha g_\beta \quad \text{and} \quad F \diamond G = w^* - \lim_{\beta} w^* - \lim_{\alpha} f_\alpha g_\beta.$$

The Banach space \mathcal{A}^{**} equipped with either of the multiplications \square or \diamond forms a Banach algebra. The Banach algebra \mathcal{A} is called *Arens regular* if two actions \square and \diamond coincide.

Let $c_0(H, \omega^{-1}) := \{f : H \rightarrow \mathbb{C} : f\omega^{-1} \in c_0(H)\}$. Note that $\ell^1(H, \omega)$ is the dual of $c_0(H, \omega^{-1})$. Hence, $\ell^1(H, \omega)^{**}$ can be decomposed as $\ell^1(H, \omega) \oplus c_0(H, \omega^{-1})^\perp$ when $c_0(H, \omega^{-1})^\perp := \{F \in \ell^1(H, \omega)^{**} : \langle F, \phi \rangle = 0 \text{ for all } \phi \in c_0(H, \omega^{-1})\}$. To see this decomposition, let $F \in \ell^1(H, \omega)^{**}$, it is clear that $f := F|_{c_0(H, \omega^{-1})} \in \ell^1(H, \omega)$ and consequently $\Phi := F - f \in c_0(H, \omega^{-1})^\perp$. Therefore, $F = (f, \Phi) \in \ell^1(H, \omega) \oplus c_0(H, \omega^{-1})^\perp$.

Proposition 4.1 *Let (H, ω) be a weighted hypergroup. Then $\ell^1(H, \omega)$ is Arens regular if the multiplications \square and \diamond restricted to $c_0(H, \omega^{-1})^\perp$ are constantly 0.*

Proof. Now let $F = (f, \Phi)$ and $G = (g, \Psi)$ belong to $\ell^1(H, \omega)^{**}$. First, note that $f\square\Psi = f\diamond\Psi$ and $\Phi\square g = \Phi\diamond g$. Thus $F\square G = (f, \Phi)\square(g, \Psi) = (fg, f\square\Psi + \Phi\square g) = (fg, f\diamond\Psi + \Phi\diamond g) = F\diamond G$. \square

Let us define the bounded function $\Omega_\omega : H \times H \rightarrow (0, 1]$ by

$$\Omega_\omega(x, y) := \frac{\omega(\delta_x * \delta_y)}{\omega(x)\omega(y)} \quad (x, y \in H). \quad (4.1)$$

If there is no risk of confusion, we may use Ω instead of Ω_ω .

For a weighted group (G, σ) , the Arens regularity of weighted group algebras has been characterized completely; [9, Theorem 8.11] proves that it is equivalent to the *0-clusterness* of the function Ω_σ on $G \times G$, that is

$$\lim_n \lim_m \Omega_\sigma(x_m, y_n) = \lim_m \lim_n \Omega_\sigma(x_m, y_n) = 0$$

whenever (x_m) and (y_n) are sequences in G , each consisting of distinct points, and both repeated limits exist. A stronger version of 0-clusterness is called *strong 0-clusterness* (see [9, Section 3]). We define strongly 0-cluster functions as presented in [9, Definition 3.6] for discrete topological spaces.

Definition 4.2 Let X and Y be two sets and f is a bounded function on $X \times Y$ into \mathbb{C} . Then f *0-clusters strongly* on $X \times Y$ if

$$\lim_{x \rightarrow \infty} \limsup_{y \rightarrow \infty} f(x, y) = \lim_{y \rightarrow \infty} \limsup_{x \rightarrow \infty} f(x, y) = 0.$$

Let us define Banach space isomorphism $\kappa : \ell^1(H, \omega) \rightarrow \ell^1(H)$ where $\kappa(f) = f\omega$ for each $f \in \ell^1(H, \omega)$. Note that for $\kappa^{**} : \ell^1(H, \omega)^{**} \rightarrow \ell^1(H)^{**}$ and $\Phi \in c_0(H, \omega)^\perp$, one gets $\langle \kappa^{**}(\Phi), \phi \rangle = \langle \Phi, \kappa^*(\phi) \rangle$ which is 0 for all $\phi \in c_0(H)$. Therefore $\kappa^{**}(\Phi) \in c_0(H)^\perp$. The converse (which we do not use here) is also true and straightforward to show.

The following theorem is a generalization of [9, Theorem 8.8]. In the proof we use some techniques of the proof of [18, Theorem 3.16].

Theorem 4.3 *Let (H, ω) be a weighted hypergroup and let Ω 0-cluster strongly on $H \times H$. Then $\Phi \square \Psi = 0$ and $\Phi \diamond \Psi = 0$ whenever $\Phi, \Psi \in c_0(H, 1/\omega)^\perp$.*

Proof. Let us show the theorem for $\Phi \square \Psi$, the proof for the other action is similar. Let $\Phi, \Psi \in c_0(H, 1/\omega)^\perp$. By Goldstine's theorem, there are nets $(f_\alpha)_\alpha, (g_\beta)_\beta \subseteq \ell^1(H)$ such that $f_\alpha \rightarrow \kappa^{**}(\Phi)$ and $g_\beta \rightarrow \kappa^{**}(\Psi)$ in the weak* topology of $\ell^1(H)^{**}$ while $\sup_\alpha \|f_\alpha\|_1 \leq 1$ and $\sup_\beta \|g_\beta\|_1 \leq 1$. So for each $\psi \in \ell^\infty(H)$ and $\Phi, \Psi \in \ell^1(H, \omega)^{**}$,

$$\langle \psi \omega, \kappa^{**}(\Phi \square \Psi) \rangle = \langle \kappa^*(\psi), \Phi \square \Psi \rangle = \lim_\alpha \lim_\beta \langle \psi \omega, \kappa^{-1}(f_\alpha) * \kappa^{-1}(g_\beta) \rangle = \lim_\alpha \lim_\beta \langle \psi \omega, f_\alpha / \omega * g_\beta / \omega \rangle.$$

Thus

$$\begin{aligned} |\langle \psi \omega, \kappa^{**}(\Phi \square \Psi) \rangle| &= \lim_\alpha \lim_\beta |\langle \psi \omega, f_\alpha / \omega * g_\beta / \omega \rangle| \\ &= \lim_\alpha \lim_\beta \left| \sum_{y \in H} \psi(y) \omega(y) \sum_{x, z \in H} \frac{f_\alpha(x)}{\omega(x)} \frac{g_\beta(z)}{\omega(z)} \delta_x * \delta_z(y) \right| \\ &\leq \lim_\alpha \sup_\beta \sum_{x, z \in H} \frac{|f_\alpha(x)|}{\omega(x)} \frac{|g_\beta(z)|}{\omega(z)} \sum_{y \in H} |\psi(y)| \omega(y) \delta_x * \delta_z(y) \\ &\leq \|\psi\|_{\ell^\infty(H)} \lim_\alpha \sup_\beta \sum_{x, z \in H} |f_\alpha(x)| |g_\beta(z)| \sum_{y \in H} \frac{\omega(y)}{\omega(x) \omega(z)} \delta_x * \delta_z(y) \\ &= \|\psi\|_{\ell^\infty(H)} \lim_\alpha \sup_\beta \sum_{x, z \in H} |f_\alpha(x)| |g_\beta(z)| \Omega(x, z). \end{aligned}$$

For a given $\epsilon > 0$, since by the hypothesis $\lim_x \limsup_z \Omega(x, z) = 0$, there is a finite set $A \subseteq H$ such that for each $x \in A^c (= H \setminus A)$ there exists a finite set $B_x \subseteq H$ such that for each $z \in B_x^c := H \setminus B_x$, $|\Omega(x, z)| \leq \epsilon$. First note that

$$\lim_\alpha \sup_\beta \sum_{x \in A^c} \sum_{z \in B_x^c} |f_\alpha(x)| |g_\beta(z)| \Omega(x, z) \leq \lim_\alpha \sup_\beta \epsilon \|f_\alpha\|_1 \|g_\beta\|_1 \leq \epsilon.$$

Also according to our assumption about Φ and Ψ and since for each $x \in H$, $\delta_x \in c_0(H, 1/\omega)$, $\lim_\alpha f_\alpha(x) = 0$ and $\lim_\beta g_\beta(x) = 0$. So for the given $\epsilon > 0$, there is α_0 such that for all $\alpha \preceq \alpha_0$, $|f_\alpha(x)| < \epsilon/|A|$ for all $x \in A$. Moreover, for each $x \in A^c$ there is some β_0^x such that for all β where $\beta_0^x \preceq \beta$, $|g_\beta(z)| < \epsilon/|B_x|$ for all $z \in B_x$ (this is possible since A and B_x are finite). Therefore, since $|\Omega(x, z)| \leq 1$,

$$\lim_\alpha \sup_\beta \sum_{x \in A} \sum_{z \in H} |f_\alpha(x)| |g_\beta(z)| \Omega(x, z) \leq \lim_\beta \epsilon \|g_\beta\|_1 = \epsilon$$

and

$$\begin{aligned} \lim_\alpha \sup_\beta \sum_{x \in A^c} \sum_{z \in B_x} |f_\alpha(x)| |g_\beta(z)| \Omega(x, z) &\leq \lim_\alpha \sum_{x \in A^c} |f_\alpha(x)| \lim_\beta \sum_{z \in B_x} |g_\beta(z)| \\ &\leq \lim_\alpha \epsilon \|f_\alpha\|_1 = \epsilon. \end{aligned}$$

But

$$\begin{aligned}
\sum_{x,z \in H} |f_\alpha(x)| |g_\beta(z)| \Omega(x, z) &= \sum_{x \in A^c, z \in B_x^c} |f_\alpha(x)| |g_\beta(z)| \Omega(x, z) \\
&+ \sum_{x \in A, z \in H} |f_\alpha(x)| |g_\beta(z)| \Omega(x, z) \\
&+ \sum_{x \in A^c, z \in B_x} |f_\alpha(x)| |g_\beta(z)| \Omega(x, z),
\end{aligned}$$

and so, one gets that $|\langle \psi \omega, \kappa^{**}(\Phi \square \Psi) \rangle| \leq 3\epsilon \|\psi\|_\infty$. Since $\epsilon > 0$ was arbitrary, this proves the claim of the theorem. \square

Theorem 4.4 *Let (H, ω) be a discrete weighted hypergroup and consider the following conditions:*

- (1) Ω 0-clusters strongly on $H \times H$.
- (2) $\Phi \square \Psi = \Phi \diamond \Psi = 0$ for all $\Phi, \Psi \in c_0(H, 1/\omega)^\perp$.
- (3) $\ell^1(H, \omega)$ is Arens regular.

Then (1) \Rightarrow (2) \Rightarrow (3).

Proof. (1) \Rightarrow (2) by Theorem 4.3. (2) \Rightarrow (3) is implied from Proposition 4.1. \square

Remark 4.5 Since in hypergroups, the cancellation does not necessarily exist, the argument of [8, Theorem 1] cannot be applied to show (3) implies (1).

Example 4.6 Let \mathbb{N}_0 be equipped with Chebyshev polynomial hypergroup structure of the first type and σ_f be the group weight defined in Example 3.17 for an increasing function f . One can easily check that if $\lim_{n,m} f(n+m)/f(n)f(m) = 0$, then Ω_{ω_f} 0-clusters strongly on $\mathbb{N}_0 \times \mathbb{N}_0$; hence, $\ell^1(\mathbb{N}_0, \omega_f)$ is Arens regular. Indeed, $Z_\pm A(\mathbb{T}, \sigma_f)$ is Arens regular. But note that $A(\mathbb{T}, \sigma_f)$ (which is isomorphic to $\ell^1(\mathbb{Z}, \sigma_f)$ through the Fourier transform) is not Arens regular, as Ω_{σ_f} does not 0-cluster strongly on $\mathbb{Z} \times \mathbb{Z}$ (see [9, Theorem 8.11]).

Corollary 4.7 *Let (H, ω) be a weighted discrete hypergroup such that ω is a weakly additive weight. If $1/\omega \in c_0(H)$, then $\ell^1(H, \omega)$ is Arens regular.*

Proof. We have

$$\begin{aligned}
\lim_{x \rightarrow \infty} \limsup_{y \rightarrow \infty} \frac{\omega(\delta_x * \delta_y)}{\omega(x)\omega(y)} &\leq \limsup_{x \rightarrow \infty} \limsup_{y \rightarrow \infty} C \frac{\omega(x) + \omega(y)}{\omega(x)\omega(y)} \\
&= C \limsup_{x \rightarrow \infty} \limsup_{y \rightarrow \infty} \frac{1}{\omega(x)} + \frac{1}{\omega(y)} = 0.
\end{aligned}$$

Therefore Ω 0-clusters strongly on $H \times H$ and hence $\ell^1(H, \omega)$ is Arens regular by Theorem 4.4. \square

Corollary 4.8 *Let H be a finitely generated hypergroup. Then for each polynomial weight ω_β ($\beta > 0$) on H defined in Definition 3.3, $\ell^1(H, \omega_\beta)$ is Arens regular.*

Proof. Here we only need to prove the case for an infinite hypergroup H . Let F be a finite generator of the hypergroup H containing the identity of H rendering the central weight ω_β . Recall that ω_β is weakly additive with constant $C = \min\{1, 2^{\beta-1}\}$. Moreover, for each $N \in \mathbb{N}$, for $x \in H \setminus F^{*N}$, $\tau_F(x) \geq N$; hence, $\omega_\beta(x) = (1 + \tau_F(x))^\beta \geq (1 + N)^\beta$. Therefore, $1/\omega_\beta \in c_0(H)$. Subsequently, $\ell^1(H, \omega_\beta)$ is Arens regular, by Corollary 4.7. \square

Remark 4.9 Every finitely generated hypergroup H admits a weight for which the corresponding weighted algebra is Arens regular. On the other hand, an argument similar to [8, Corollary 1] may apply to show that for every uncountable discrete hypergroup H , H does not have any weight ω which 0-clusters strongly.

Example 4.10 Let ω_β be as defined in Lemma 3.15 for some $\beta \geq 0$. Then Ω_{ω_β} also 0-clusters strongly on $\widehat{\text{SU}}(2) \times \widehat{\text{SU}}(2)$. Therefore, $\ell^1(\widehat{\text{SU}}(2), \omega_\beta)$, which is isometrically Banach algebra isomorphic to $ZA(\text{SU}(2), \omega_\beta)$, is Arens regular. On the other hand, $A(\text{SU}(2), \omega_\beta)$ is not Arens regular if $\beta > 0$. To observe the later fact, first note that by applying [8], we obtain that $\ell^1(\mathbb{Z}, \sigma)$ is not Arens regular for σ defined in (3.4). Therefore, $A(\mathbb{T}, \sigma)$ is not Arens regular. Note that, ω_β can also be rendered using the weight σ through the argument of the last paragraph of Subsection 3.3. For the dual spaces $VN(\mathbb{T}, \sigma)$ and $VN(\text{SU}(2), \omega_\beta)$, one may verify that $VN(\mathbb{T}, \sigma)$ embeds $*$ -weakly in $VN(\text{SU}(2), \omega_\beta)$ (the details of this embedding will appear in a manuscript by the second named author and et al). Hence, $A(\mathbb{T}, \sigma)$ is a quotient of $A(\text{SU}(2), \omega_\beta)$ and consequently $A(\text{SU}(2), \omega_\beta)$ is not Arens regular.

In the following, we generalize some results on $\text{SU}(2)$ to all $\text{SU}(n)$'s, the group of all $n \times n$ special unitary matrices on \mathbb{C} , based on a recent study on the representation theory of $\text{SU}(n)$, [7]. As an example for Lemma 3.15, $(\widehat{\text{SU}}(n), \omega_\beta)$ is a discrete commutative hypergroup where $\omega_\beta(\pi) = d_\pi^\beta$ for some $\beta \geq 0$. See [10] for the details of representation theory of $\text{SU}(n)$. There is a one-to-one correspondence between $\widehat{\text{SU}}(n)$ and n -tuples $(\pi_1, \dots, \pi_n) \in \mathbb{N}_0^n$ such that $\pi_1 \geq \pi_2 \geq \dots \geq \pi_{n-1} \geq \pi_n = 0$. This presentation of the representation theory of $\text{SU}(n)$ is called *dominant weight*. Using this presentation, we have the following formula which gives the dimension of each representation by the formula

$$d_\pi = \prod_{1 \leq i < j \leq n} \frac{\pi_i - \pi_j + j - i}{j - i} \quad (4.2)$$

where π is the representation corresponding to (π_1, \dots, π_n) . Suppose that π, ν, μ are representations corresponding to (π_1, \dots, π_n) , (ν_1, \dots, ν_n) , and (μ_1, \dots, μ_n) , respectively, such that $\pi \in \text{supp}(\delta_\nu * \delta_\mu)$. Collins, Lee, and Śniady showed in [7, Corollary 1.2] there exists some $C_n > 0$, for each $n \in \mathbb{N}$, such that

$$\frac{d_\pi}{d_\mu d_\nu} \leq C_n \left(\frac{1}{1 + \mu_1} + \frac{1}{1 + \nu_1} \right). \quad (4.3)$$

Applying (4.3), we prove that ω_β 0-clusters on $\widehat{\text{SU}}(n)$.

Proposition 4.11 *For every $\beta > 0$, $\ell^1(\widehat{\text{SU}}(n), \omega_\beta)$ is Arens regular.*

Proof. Let $(\mu_m)_{m \in \mathbb{N}}$ and $(\nu_k)_{k \in \mathbb{N}}$ be two arbitrary sequences of distinct elements of $\widehat{\text{SU}}(n)$. Since, the elements of $(\mu_m)_{m \in \mathbb{N}}$ are distinct, $\lim_{m \rightarrow \infty} \mu_1^{(m)} = \infty$ where $\mu_m = (\mu_1^{(m)}, \dots, \mu_n^{(m)})$. The very same thing can be said for $\nu_k = (\nu_1^{(k)}, \dots, \nu_n^{(k)})$. For each arbitrary pair $(m, k) \in \mathbb{N} \times \mathbb{N}$, if $\pi \in \text{supp}(\delta_{\mu_m} * \delta_{\nu_k})$, we have

$$d_\pi \leq C_n \left(\frac{1}{1 + \mu_1^{(m)}} + \frac{1}{1 + \nu_1^{(k)}} \right) d_{\mu_m} d_{\nu_k}.$$

Hence

$$\omega_\beta(\pi) \leq C_n^\beta \left(\frac{1}{1 + \mu_1^{(m)}} + \frac{1}{1 + \nu_1^{(k)}} \right)^\beta \omega_\beta(\mu_m) \omega_\beta(\nu_k).$$

Therefore

$$\omega_\beta(\delta_{\mu_m} * \delta_{\nu_k}) = \sum_{\pi \in \widehat{\text{SU}}(n)} \delta_{\mu_m} * \delta_{\nu_k}(\pi) \omega_\beta(\pi) \leq C_n^\beta \left(\frac{1}{1 + \mu_1^{(m)}} + \frac{1}{1 + \nu_1^{(k)}} \right)^\beta \omega_\beta(\mu_m) \omega_\beta(\nu_k).$$

Or equivalently

$$\Omega_\beta(\mu_m, \nu_k) := \frac{\omega_\beta(\delta_{\mu_m} * \delta_{\nu_k})}{\omega_\beta(\mu_m) \omega_\beta(\nu_k)} \leq C_n^\beta \left(\frac{1}{1 + \mu_1^{(m)}} + \frac{1}{1 + \nu_1^{(k)}} \right)^\beta.$$

Hence, $\lim_{m \rightarrow \infty} \limsup_{k \rightarrow \infty} \Omega_\beta(\mu_m, \nu_k) = \lim_{k \rightarrow \infty} \limsup_{m \rightarrow \infty} \Omega_\beta(\mu_m, \nu_k) = 0$. Since $\widehat{\text{SU}}(n)$ is countable, this argument implies that Ω_β 0-clusters strongly on $\widehat{\text{SU}}(n) \times \widehat{\text{SU}}(n)$ and, by Theorem 4.4, $\ell^1(\widehat{\text{SU}}(n), \omega_\beta)$ is Arens regular. \square

Example 4.12 Let $SL(2, 2^n)$ denote the finite group of special linear matrices over the field \mathbb{F}_{2^n} with cardinal 2^n , for given $n \in \mathbb{N}$. As a direct result of the character table, [1], for each three conjugacy classes say $C_1, C_2, D \in \text{Conj}(SL(2, 2^n))$, $|D| \leq 2(|C_1| + |C_2|)$ if $D \subseteq C_1 C_2$ for all n . Let us define the FC group G to be the restricted direct product of $\{SL(2, 2^n)\}_{n \in \mathbb{N}}$ i.e. $G := \bigoplus_{n=1}^{\infty} SL(2, 2^n)$. Therefore, one can easily show that the weight ω_α , defined in Example 3.8, is a weakly additive weight with the constant $M = 2^\alpha \min\{1, 2^{\alpha-1}\}$. Moreover, since $\lim_{C \rightarrow \infty} \omega_\alpha(C) = \infty$, $\ell^1(\text{Conj}(G), \omega_\alpha)$ is Arens regular, by Corollary 4.7.

Remark 4.13 Let ω be a central weight on $\text{Conj}(G)$ for some FC group G . Then there is a group weight σ_ω , as defined in Remark 3.6, such that $\ell^1(\text{Conj}(G), \omega)$ is isometrically Banach algebra isomorphic to $Z\ell^1(G, \sigma_\omega)$. So one may also use the embedding $\ell^1(\text{Conj}(G), \omega) \hookrightarrow \ell^1(G, \sigma_\omega)$ to study Example 4.12 by applying the theorems which are characterizing Arens regularity of weighted group algebras.

Remark 4.14 Let G be an FC group and σ a group weight on G . We defined ω_σ , the derived weight on $\text{Conj}(G)$ from σ in Proposition 3.5. Recall that in this case $Z\ell^1(G, \sigma)$ is isomorphic to the Banach algebra $\ell^1(\text{Conj}(G), \omega_\sigma)$. If N is a normal subgroup of G , we defined a quotient mapping $T_{\omega_\sigma} : \ell^1(\text{Conj}(G), \omega_\sigma) \rightarrow \ell^1(\text{Conj}(G/N), \tilde{\omega}_\sigma)$ in Corollary 3.12 where $\tilde{\omega}_\sigma(C_{xN}) = \inf\{\omega_\sigma(C_{xy}) : y \in N\}$ ($C_{xN} \in \text{Conj}(G/N)$). Let us note that for an Arens regular Banach algebra \mathcal{A} , every quotient algebra \mathcal{A}/\mathcal{I} where \mathcal{I} is a closed ideal of \mathcal{A} is Arens regular as well (see [9, Corollary 3.15]). Therefore, if $\ell^1(\text{Conj}(G), \omega_\sigma)$ is Arens regular, for every normal subgroup N , $\ell^1(\text{Conj}(G/N), \tilde{\omega}_\sigma)$, which is isomorphic to $\ell^1(\text{Conj}(G), \omega_\sigma)/\text{Ker}(T_{\omega_\sigma})$, is Arens regular.

In the final result of this section, we apply some techniques of [8] to show that for restricted direct product of hypergroups, product weights fail to admit Arens regular algebras.

Proposition 4.15 *Let $\{H_i\}_{i \in \mathbf{I}}$ be an infinite family of non-trivial discrete hypergroups and for each $i \in \mathbf{I}$, ω_i is a weight on H_i such that $\omega_i(e_{H_i}) = 1$ for all except finitely many $i \in \mathbf{I}$. Let $H = \bigoplus_{i \in \mathbf{I}} H_i$ and $\omega = \prod_{i \in \mathbf{I}} \omega_i$. Then $\ell^1(H, \omega)$ is not Arens regular.*

Proof. Since \mathbf{I} is infinite, suppose that $\mathbb{N}_0 \times \mathbb{N}_0 \subseteq \mathbf{I}$. Define $v_n = (x_i)_{i \in \mathbf{I}}$ where $x_i = e_{H_i}$ for all $i \in \mathbf{I} \setminus (n, 0)$ and $x_{(n,0)}$ be a non-identity element of $H_{(n,0)}$ for all $n \in \mathbb{N}$. Similarly define $u_m = (x_i)_{i \in \mathbf{I}}$ where $x_i = e_{H_i}$ for all $i \in \mathbf{I} \setminus (0, m)$ and $x_{(0,m)}$ be a non-identity element of $H_{(0,m)}$ for all $m \in \mathbb{N}$. Note that for each pair of elements $(n, m) \in \mathbb{N} \times \mathbb{N}$, $\text{supp}(\delta_{v_n} * \delta_{u_m})$ forms a singleton in H ; moreover, $\omega(\delta_{v_n} * \delta_{u_m}) = \omega(v_n)\omega(u_m)$. Hence, $(\delta_{v_n} * \delta_{u_m})_{(n,m) \in \mathbb{N} \times \mathbb{N}}$ forms a sequence of distinct elements in $\ell^1(H)$.

Let us define $f_n = \delta_{v_n}$ and $g_m = \delta_{u_m}$ for all $n, m \in \mathbb{N}$. Suppose that $A := \{(v_n, u_m) : n > m\}$ and $\phi \in \ell^\infty(H)$ is the characteristic function of the subset A . Clearly, $\kappa^{-1}(f_n) = \omega^{-1}f_n$ and $\kappa^{-1}(g_m) = \omega^{-1}g_m$ belong to $\ell^1(H, \omega)$ for all n, m and $\kappa^*(\phi) = \omega\phi \in \ell^\infty(H, \omega^{-1})$, for the Banach space isomorphism $\kappa : \ell^1(H, \omega) \rightarrow \ell^1(H)$ where $\kappa(f) = f\omega$ for each $f \in \ell^1(H, \omega)$. Note that

$$\begin{aligned} \langle \omega^{-1}f_n * \omega^{-1}g_m, \kappa^*(\phi) \rangle &= \langle \omega^{-1}f_n * \omega^{-1}g_m, \omega\phi \rangle \\ &= \sum_{t \in H} (\omega^{-1}f_n * \omega^{-1}g_m)(t) \omega(t) \phi(t) \\ &= \frac{\omega(v_n * u_m)}{\omega(v_n)\omega(u_m)} \phi(\delta_{v_n} * \delta_{u_m}) \\ &= \phi(\delta_{v_n} * \delta_{u_m}) = \begin{cases} 1 & \text{if } n > m \\ 0 & \text{if } n \leq m \end{cases} \end{aligned}$$

Let us recall that for each n and m , $\|f_n\|_{\ell^1(H, \omega)} = 1$ and $\|g_m\|_{\ell^1(H, \omega)} = 1$. So $(f_n)_{n \in \mathbb{N}}$ and $(g_m)_{m \in \mathbb{N}}$, as two nets in the unit ball of $\ell^1(H, \omega)^{**}$, have two subnet $(f_\alpha)_\alpha$ and $(g_\beta)_\beta$ such that f_α and g_β converge weakly* to some F and G in $\ell^1(H, \omega)^{**}$, respectively. Note that for the specific element ϕ , defined above, $\langle F \square G, \phi \rangle = 0$ while $\langle F \diamond G, \phi \rangle = 1$. Hence $F \square G \neq F \diamond G$ and consequently $\ell^1(H, \omega)$ is not Arens regular. \square

5 Isomorphism to operator algebras

Let (H, ω) be a weighted discrete hypergroup. In this section, we study the existence of an algebra isomorphism from $\ell^1(H, \omega)$ onto an operator algebra. A Banach algebra \mathcal{A} is called an operator algebra if there is a Hilbert space \mathcal{H} such that \mathcal{A} is a closed subalgebra of $\mathcal{B}(\mathcal{H})$. Let \mathcal{A} be a Banach algebra and $\mathbf{m} : \mathcal{A} \times \mathcal{A} \rightarrow \mathcal{A}$ is the bilinear (multiplication) mapping $\mathbf{m}(f, g) = fg$. Then \mathcal{A} is called *injective*, if \mathbf{m} has a bounded extension from $\mathcal{A} \otimes_\epsilon \mathcal{A}$ into \mathcal{A} , where \otimes_ϵ is the injective tensor product. In this case, we denote the norm of \mathbf{m} by $\|\mathbf{m}\|_\epsilon$. [19, Corollary 2.2.] proves that if a Banach algebra \mathcal{A} is injective then it is isomorphic to an operator algebra. But the converse also holds for weighted hypergroup algebras. The proof is similar to the group case in [19, Theorem 2.8] and it follows from the little Grothendieck inequality (see [21]). Note that a Banach algebra which is isomorphic to an operator algebra is always Arens regular ([5, Corollary 2.5.4]).

Injectivity of weighted group algebras has been studied before. Initially Varopoulos, in [24], studied the group \mathbb{Z} equipped with the weight $\sigma_\alpha(n) = (1 + |n|)^\alpha$ for all $\alpha \geq 0$. This study looked at injectivity of $\ell^1(\mathbb{Z}, \sigma_\alpha)$. He showed that $\ell^1(\mathbb{Z}, \sigma_\alpha)$ is injective if and only if $\alpha > 1/2$. The manuscript [19], which studied the injectivity question for a wider family of weighted group algebras, developed a machinery applying Littlewood multipliers. In particular, it partially extended Varopoulos's result to finitely generated groups with polynomial growth. Following the structure of [19], in this section, we study the injectivity or equivalently isomorphism to operator algebras for weighted hypergroup algebras.

In this section, $\mathcal{A} \otimes_\gamma \mathcal{B}$ and $\mathcal{A} \otimes_\epsilon \mathcal{B}$ denote respectively the projective and injective tensor products of Banach spaces \mathcal{A} and \mathcal{B} .

We know that $\ell^1(H, \omega) \otimes_\gamma \ell^1(H, \omega)$ is isometrically isomorphic to $\ell^1(H \times H, \omega \times \omega)$. Moreover, $\ell^1(H \times H, \omega \times \omega)^*$ is $\ell^\infty(H \times H, \omega^{-1} \times \omega^{-1})$. Since the injective tensor norm is minimal among all cross-norm Banach space tensor norms, the identity map $\iota : \ell^1(H) \times \ell^1(H) \rightarrow \ell^1(H) \otimes_\epsilon \ell^1(H)$ may extend to a contractive mapping

$$\iota : \ell^1(H) \otimes_\gamma \ell^1(H) \rightarrow \ell^1(H) \otimes_\epsilon \ell^1(H).$$

Since, ι has a dense range,

$$\iota^* : (\ell^1(H) \otimes_\epsilon \ell^1(H))^* \rightarrow (\ell^1(H) \otimes_\gamma \ell^1(H))^* = \ell^\infty(H \times H) \quad (5.1)$$

is an injective mapping. Therefore, applying ι^* , one may embed $(\ell^1(H) \otimes_\epsilon \ell^1(H))^*$ into $\ell^\infty(H \times H)$, as a linear subspace of $\ell^\infty(H \times H)$.

Let H be a discrete hypergroup. We define *Littlewood multipliers* of H to be the set of all functions $f : H \times H \rightarrow \mathbb{C}$ such that there exist functions $f_1, f_2 : H \times H \rightarrow \mathbb{C}$ where $f(x, y) = f_1(x, y) + f_2(x, y)$ for $x, y \in G$ such that

$$\sup_{y \in H} \sum_{x \in H} |f_1(x, y)|^2 < \infty \quad \text{and} \quad \sup_{x \in H} \sum_{y \in H} |f_2(x, y)|^2 < \infty.$$

We denote the set of all Littlewood multipliers by $T^2(H)$ and define the norm $\|\cdot\|_{T^2(H)}$ by

$$\|f\|_{T^2(H)} := \inf \left\{ \sup_{y \in H} \left(\sum_{x \in H} |f_1(x, y)|^2 \right)^{1/2} + \sup_{x \in H} \left(\sum_{y \in H} |f_2(x, y)|^2 \right)^{1/2} \right\}$$

where the infimum is taken over all possible decompositions f_1, f_2 . Note that for a decomposition f_1, f_2 of $f \in T^2(H)$,

$$\begin{aligned} \|f\|_{\ell^\infty(H \times H)} = \sup_{x, y \in H} |f(x, y)| &\leq \sup_{x, y \in H} |f_1(x, y)| + \sup_{x, y \in H} |f_2(x, y)| \\ &\leq \sup_{y \in H} \left(\sum_{x \in H} |f_1(x, y)|^2 \right)^{1/2} + \sup_{x \in H} \left(\sum_{y \in H} |f_2(x, y)|^2 \right)^{1/2} < \infty, \end{aligned}$$

since for discrete space H , $\ell^2(H) \subseteq \ell^\infty(H)$ and $\|\cdot\|_\infty \leq \|\cdot\|_2$. Since f_1, f_2 , in the previous equation are arbitrary, $\|f\|_{\ell^\infty(H \times H)} \leq \|f\|_{T^2(H)}$. Hence $T^2(H) \subseteq \ell^\infty(H \times H)$. Furthermore, for each $\phi \in \ell^\infty(H \times H)$ and $f \in T^2(H)$, $f\phi \in T^2(H)$ and $\|f\phi\|_{T^2(H)} \leq \|f\|_{T^2(H)} \|\phi\|_\infty$.

The following theorem is the hypergroup version of [19, Theorem 2.7]. Since the proof is very similar to the group case, we omit it here (although with all the details it can be found in [2]). Here we use $\mathcal{K}_{\mathfrak{G}}$ to denote *Grothendieck's constant*. First in his celebrated “Résumé”, Grothendieck proved the existence of the constant $\mathcal{K}_{\mathfrak{G}}$ in Grothendieck's inequality. For a detailed account of Grothendieck's constant, its history, and approximations look at [21, Sections 3 and 4].

Theorem 5.1 *Let $I : T^2(H) \rightarrow (\ell^1(H) \otimes_\gamma \ell^1(H))^* = \ell^\infty(H \times H)$ be the mapping which takes every element of $T^2(H)$ to itself as a bounded function on $H \times H$. Then $I(T^2(H)) \subseteq \iota^*((\ell^1(H) \otimes_\epsilon \ell^1(H))^*)$ for the mapping ι^* defined in (5.1). Moreover, $J := \iota^{*-1} \circ I : T^2(H) \rightarrow (\ell^1(H) \otimes_\epsilon \ell^1(H))^*$ is bounded and $\|J\| \leq \mathcal{K}_{\mathfrak{G}}$.*

From now on, we identify $(\ell^1(H) \otimes_\epsilon \ell^1(H))^*$ with its image through the mapping ι^* ; hence, J is the identity mapping which takes $T^2(H)$ into $(\ell^1(H) \otimes_\epsilon \ell^1(H))^*$. We present our first main result of this section. This is a generalization of [19, Theorem 3.1].

Theorem 5.2 *Let H be a discrete hypergroup and ω is a weight on H such that Ω , defined in (4.1), belongs to $T^2(H)$. Then $\ell^1(H, \omega)$ is injective and equivalently isomorphic to an operator algebra. Moreover, for the multiplication map \mathbf{m} on $\ell^1(H, \omega) \otimes_\epsilon \ell^1(H, \omega)$, as defined before, $\|\mathbf{m}\|_\epsilon \leq \mathcal{K}_{\mathfrak{G}} \|\Omega\|_{T^2(H)}$.*

Proof. Let $\Gamma_\omega : \ell^1(H \times H, \omega \times \omega) \rightarrow \ell^1(H, \omega)$ such that $\Gamma_\omega(f \otimes g) := f * g$ for $f, g \in \ell^1(H, \omega)$. The adjoint of Γ_ω , Γ_ω^* , can be characterized as follows.

$$\Gamma_\omega^*(\phi)(x, y) = \langle \Gamma_\omega^*(\phi), \delta_x \otimes \delta_y \rangle = \langle \phi, \Gamma_\omega(\delta_x \otimes \delta_y) \rangle = \langle \phi, \delta_x * \delta_y \rangle$$

for all $\phi \in \ell^\infty(H, \omega^{-1})$ and $x, y \in H$. Now we define L from $\ell^\infty(H)$ to $\ell^\infty(H \times H)$ such that the following diagram commutes,

$$\begin{array}{ccc} \ell^\infty(H, \omega^{-1}) & \xrightarrow{\Gamma_\omega^*} & \ell^\infty(H \times H, \omega^{-1} \times \omega^{-1}) \\ P \uparrow & & \downarrow R \\ \ell^\infty(H) & \xrightarrow{L} & \ell^\infty(H \times H) \end{array}$$

where $P(\varphi)(x) = \varphi(x)\omega(x)$ for $\varphi \in \ell^\infty(H)$ and $R(\phi)(x, y) = \phi(x, y)\omega^{-1}(x)\omega^{-1}(y)$ for $\phi \in \ell^\infty(H \times H, \omega^{-1} \times \omega^{-1})$ and $x, y \in H$. Hence, one gets

$$\begin{aligned} L(\varphi)(x, y) &= R(\Gamma_\omega^* \circ P(\varphi))(x, y) = \frac{(\Gamma_\omega^* \circ P(\varphi))(x, y)}{\omega(x)\omega(y)} \\ &= \frac{\Gamma_\omega^*(\omega\varphi)(x, y)}{\omega(x)\omega(y)} \\ &= \frac{\langle \varphi\omega, \delta_x * \delta_y \rangle}{\omega(x)\omega(y)} \\ &= \sum_{t \in H} \delta_x * \delta_y(t) \frac{\omega(t)}{\omega(x)\omega(y)} \varphi(t). \end{aligned}$$

for all $\varphi \in \ell^\infty(H)$. Hence,

$$\left| \sum_{t \in H} \delta_x * \delta_y(t) \frac{\omega(t)}{\omega(x)\omega(y)} \varphi(t) \right| \leq \sum_{t \in H} \delta_x * \delta_y(t) \frac{\omega(t)}{\omega(x)\omega(y)} |\varphi(t)| \leq \|\varphi\|_\infty \Omega(x, y)$$

So there is a function $v_\varphi : H \times H \rightarrow \mathbb{C}$ such that

$$\frac{\langle \delta_x * \delta_y, \omega\varphi \rangle}{\omega(x)\omega(y)} = v_\varphi(x, y) \|\varphi\|_\infty \Omega(x, y)$$

and $\|v_\varphi\|_\infty \leq 1$. Therefore $L(\varphi) = \Lambda(\varphi)\Omega$ where $\Lambda(\varphi)(x, y) := v_\varphi(x, y)\|\varphi\|_\infty$ for all $\varphi \in \ell^\infty(H)$. Since Ω belongs to $T^2(H)$ and $T^2(H)$ is an $\ell^\infty(H \times H)$ -module, $L(\varphi) \in T^2(H)$ and $\|L(\varphi)\|_{T^2(H)} \leq \|\varphi\|_\infty \|\Omega\|_{T^2(H)}$. Therefore $L(\ell^\infty(H)) \subseteq T^2(H) \subseteq (\ell^1(H) \otimes_\epsilon \ell^1(H))^*$.

In this case, using the following diagram with $\mathcal{A} = R^{-1}((\ell^1(H) \otimes_\epsilon \ell^1(H))^*)$,

$$\begin{array}{ccccc} \ell^\infty(H, \omega^{-1}) & \xrightarrow{\Gamma_\omega^*} & \mathcal{A} & \xrightarrow{\iota} & \ell^\infty(H \times H, \omega^{-1} \times \omega^{-1}) \\ P \uparrow & & \downarrow R|_r & & \downarrow R \\ \ell^\infty(H) & \xrightarrow{L} & (\ell^1(H) \otimes_\epsilon \ell^1(H))^* & \xrightarrow{\iota} & \ell^\infty(H \times H) \end{array}$$

One can easily verify that $\mathcal{A} = (\ell^1(H, \omega) \otimes_\epsilon \ell^1(H, \omega))^*$. So, we have shown that Γ^* is a map projecting $\ell^\infty(H)$ into $(\ell^1(H) \otimes_\epsilon \ell^1(H))^*$ as a subset of $\ell^\infty(H \times H)$. we see that Γ_ω^* is a map

projecting $\ell^\infty(H, \omega^{-1})$ into $(\ell^1(H, \omega) \otimes_\epsilon \ell^1(H, \omega))^*$. Hence, $\Gamma_\omega^* = \mathbf{m}^*$, where \mathbf{m} is the multiplication extended to $\ell^1(H, \omega) \otimes_\epsilon \ell^1(H, \omega)$. Therefore \mathbf{m} is bounded and $\|\mathbf{m}\| = \|\Gamma_\omega\| = \|R\Gamma_\omega P\| = \|L\|$. Moreover,

$$\begin{aligned} \|L(\varphi)\|_{(\ell^1(H) \otimes_\epsilon \ell^1(H))^*} &\leq \|J\| \|\Gamma^*(\varphi)\|_{T^2(H)} \leq \mathcal{K}_{\mathfrak{G}} \|\Omega\|_{T^2(H)} \|\Lambda(\varphi)\|_{\ell^\infty(H \times H)} \\ &\leq \mathcal{K}_{\mathfrak{G}} \|\Omega\|_{T^2(H)} \|\varphi\|_{\ell^\infty(H)} \end{aligned}$$

for all $\varphi \in \ell^\infty(H)$. Consequently, $\|\mathbf{m}\|_\epsilon \leq \mathcal{K}_{\mathfrak{G}} \|\Omega\|_{T^2(H)}$. \square

Example 5.3 Let ω_β be the dimension weight defined on $\widehat{\text{SU}}(n)$ in Lemma 3.15. As we have shown in the proof of Proposition 4.11, for the polynomial weight ω_β , $\beta \geq 0$, and $\mu, \nu \in \widehat{\text{SU}}(n)$,

$$\Omega_\beta(\mu, \nu) \leq C_n^\beta \left(\frac{1}{1 + \mu_1} + \frac{1}{1 + \nu_1} \right)^\beta \leq A_\beta C_n^\beta \left(\frac{1}{(1 + \mu_1)^\beta} + \frac{1}{(1 + \nu_1)^\beta} \right),$$

where $A_\beta = \min\{1, 2^{\beta-1}\}$. To study $\|\cdot\|_{T^2(\widehat{\text{SU}}(2))}$ for Ω_β , let us note that for each $k \in \mathbb{N} \cup \{0\}$, there are less than $(1 + k)^{n-2}$ many $\lambda = (\lambda_1, \dots, \lambda_n) \in \widehat{\text{SU}}(n)$ such that $\lambda_1 = k$. Therefore

$$\sum_{\lambda \in \widehat{\text{SU}}(n)} \frac{1}{(1 + \lambda_1)^{2\beta}} \leq \sum_{k=0}^{\infty} \frac{(1 + k)^{n-2}}{(1 + k)^{2\beta}}$$

where the right-hand side series converges if and only if $2\beta - n + 2 > 1$. Therefore, for $\beta > (n-1)/2$, $\Omega_\beta \in T^2(\widehat{\text{SU}}(n))$ and by Theorem 5.2, $\ell^1(\widehat{\text{SU}}(2), \omega_\beta)$ is injective and consequently isomorphic to an operator algebra. Moreover, note that

$$\begin{aligned} \|\Omega_\beta\|_{T^2(\widehat{\text{SU}}(n))} &\leq \left\| (\mu, \nu) \mapsto \frac{A_\beta C_n^\beta}{1 + \mu_1} + \frac{A_\beta C_n^\beta}{1 + \nu_1} \right\|_{T^2(H)} \\ &\leq \sup_{\nu \in \widehat{\text{SU}}(n)} \left(\sum_{\mu \in \widehat{\text{SU}}(n)} \left| \frac{A_\beta C_n^\beta}{1 + \mu_1} \right|^2 \right)^{1/2} \\ &\quad + \sup_{\mu \in \widehat{\text{SU}}(n)} \left(\sum_{\nu \in \widehat{\text{SU}}(n)} \left| \frac{A_\beta C_n^\beta}{1 + \nu_1} \right|^2 \right)^{1/2} \\ &\leq A_\beta C_n^\beta 2 \left(\sum_{k=0}^{\infty} \frac{1}{(1 + k)^{2\beta - n + 2}} \right)^{1/2}. \end{aligned}$$

Hence, for $A_\beta = \min\{1, 2^{\beta-1}\}$,

$$\|\mathbf{m}\|_\epsilon \leq 2\mathcal{K}_{\mathfrak{G}} A_\beta C_n^\beta \left(\sum_{k=0}^{\infty} \frac{1}{(1 + k)^{2\beta - n + 2}} \right)^{1/2}$$

Corollary 5.4 *Let H be a discrete hypergroup and ω is a weakly additive weight on H with a corresponding constant $C > 0$. Then $\ell^1(H, \omega)$ is injective if $\sum_{x \in H} \omega(x)^{-2} < \infty$. Moreover,*

$$\|\mathbf{m}\|_\epsilon \leq 2CK_{\mathfrak{G}} \left(\sum_{x \in H} \frac{1}{\omega(x)^2} \right)^{1/2}.$$

Proof. Suppose that $\sum_{x \in H} \omega(x)^{-2} < \infty$. Note that for each $t \in \text{supp}(\delta_x * \delta_y)$,

$$\frac{\omega(t)}{\omega(x)\omega(y)} \leq C \frac{\omega(x) + \omega(y)}{\omega(x)\omega(y)} = \frac{C}{\omega(x)} + \frac{C}{\omega(y)}.$$

Thus, for the functions $f_1(x, y) = \omega(x)^{-1}$ and $f_2(x, y) = \omega(y)^{-1}$,

$$\begin{aligned} \|\Omega\|_{T^2(H)} &\leq \left\| (x, y) \mapsto \frac{C}{\omega(x)} + \frac{C}{\omega(y)} \right\|_{T^2(H)} \\ &\leq \left(\sup_{y \in H} \left(\sum_{x \in H} \left| \frac{C}{\omega(x)} \right|^2 \right)^{1/2} + \sup_{x \in H} \left(\sum_{y \in H} \left| \frac{C}{\omega(y)} \right|^2 \right)^{1/2} \right) \leq 2C \left(\sum_{x \in H} \frac{1}{\omega(x)^2} \right)^{\frac{1}{2}}. \end{aligned}$$

Consequently, by Theorem 5.2, $\ell^1(H, \omega)$ is injective and $\|\mathbf{m}\|_\epsilon$ satisfies the mentioned inequality. \square

Example 5.5 Let ω_f be the weight constructed by the group weight admitted by a positive increasing function f (see Example 3.17). One can see that, if

$$\sum_{n \in \mathbb{N}_0} \frac{1}{f(n)^2} < \infty \quad \text{and} \quad \sup_{n, m \in \mathbb{N}_0} \frac{f(n+m)}{f(n) + f(m)} < \infty,$$

then ω_f satisfies the conditions of Corollary 5.4 and therefore, $\ell^1(\mathbb{N}_0, \omega_f)$ is isomorphic to an operator algebra. On the other hand, $\ell^1(\mathbb{N}_0, \omega_f)$ can be embedded (isomorphically as a Banach algebra) into $A(\mathbb{T}, \sigma_f)$ which is not isomorphic to any operator algebra (as it is not even Arens regular, see Example 4.6).

Remark 5.6 Note that the assumed condition for f in Example 5.5 implies the Arens regularity condition required in Example 4.6. Compare it with this known fact that every Banach algebra which is isomorphic to an operator algebra is Arens regular.

Remark 5.7 Let $(\text{Conj}(G), \omega_\alpha)$ be the weighted hypergroup defined in Example 4.12. Note that ω_α is a weakly additive weight. One can straightforwardly show that $\sum_{C \in \text{Conj}(G)} \omega(C)^{-2} = \infty$. Hence, not all weakly additive weights are satisfying the other condition mentioned in Corollary 5.4.

For finitely generated hypergroups, we showed that polynomial weights are weakly additive. In the following, we study operator algebra isomorphism for weighted hypergroup algebras with polynomial weights. Developing a machinery which relates exponential weights to polynomial ones, we also study exponential weights in Subsection 5.1. For the case that H is a group, this has been achieved in [19]

Corollary 5.8 *Let H be a finitely generated hypergroup. If F is a generator of H such that $|F^{*n}| \leq Dn^d$ for some $d, D > 0$ and ω_β is the polynomial weight on H associated to F . Then $\ell^1(H, \omega_\beta)$ is injective if $2\beta > d + 1$. Moreover, for $C = \min\{1, 2^{\beta-1}\}$,*

$$\|\mathbf{m}\|_\epsilon \leq 2CK_{\mathfrak{G}} \left(1 + \sum_{n=1}^{\infty} \frac{Dn^d}{(1+n)^{2\beta}} \right)^{1/2}.$$

Proof. To prove this corollary, we mainly rely on Corollary 5.4. Recall that ω_β is weakly additive whose constant is $C = \min\{1, 2^{\beta-1}\}$. To show the desired bound for $\|\mathbf{m}\|_\epsilon$, note that

$$\begin{aligned} \sum_{x \in H} \frac{1}{\omega_\beta(x)^2} &= \sum_{x \in H} \frac{1}{(1 + \tau(x))^{2\beta}} = \sum_{n=0}^{\infty} \sum_{\{x \in F^n \setminus F^{n-1}\}} \frac{1}{(1+n)^{2\beta}} \\ &\leq 1 + \sum_{n=1}^{\infty} \frac{|F^n|}{(1+n)^{2\beta}} \leq 1 + \sum_{n=1}^{\infty} \frac{Dn^d}{(1+n)^{2\beta}} \end{aligned}$$

which is convergent if $2\beta > d + 1$. □

Example 5.9 For a polynomial hypergroup \mathbb{N}_0 , as a finitely generated hypergroup with the generator $F = \{0, 1\}$, we have $|F^{*n}| = n + 1 \leq 2n$, as we have seen before. By Corollary 5.8, for the polynomial weight ω_β with $\beta > 1$ associated to F , $\ell^1(\mathbb{N}_0, \omega_\beta)$ is injective. For $C = \min\{1, 2^{\beta-1}\}$, Corollary 5.4 implies that

$$\|\mathbf{m}\|_\epsilon \leq 2CK_{\mathfrak{G}} \left(\sum_{n=1}^{\infty} \frac{1}{n^{2\beta}} \right)^{1/2}.$$

5.1 Hypergroups with exponential weights

The other class of weights introduced for finitely generated hypergroups is the class of exponential weights. As we mentioned before, unlike polynomial weights, exponential weights are not necessarily weakly additive. In this subsection, following [19], we study operator algebra isomorphism of these weights by studying the cases for them Ω belongs to $T^2(H)$. The following lemma is a hypergroup adaptation of [19, Theorem 3.3]. Since the proof is similar to the one of [13, Lemma B.2], we omit it here.

Lemma 5.10 Suppose that $0 < \alpha < 1$, $C > 0$, and $\beta \geq \max\left\{1, \frac{6}{C\alpha(1-\alpha)}\right\}$. Define the functions $p : [0, \infty) \rightarrow \mathbb{R}$ and $q : (0, \infty) \rightarrow \mathbb{R}$ by $p(x) := Cx^\alpha - \beta \ln(1+x)$ and $q(x) := \frac{p(x)}{x}$. Let H be a finitely generated hypergroup with a symmetric generator F and $\omega : H \rightarrow (0, \infty)$ such that

$$\omega(x) = e^{p(\tau_F(x))} = e^{\tau_F(x)q(\tau_F(x))} \quad \text{for all } x \in H.$$

Then $\omega(t) \leq M\omega(x)\omega(y)$ for all $t, x, y \in H$ such that $t \in x * y$ where

$$M = \max\{e^{p(z_1)-p(z_2)-p(z_3)} : z_1, z_2, z_3 \in [0, 2K] \cap \mathbb{N}_0\}$$

and

$$K = \left(\frac{\beta^2}{C\alpha(1-\alpha)}\right)^{1/\alpha}.$$

Theorem 5.11 Let H be a finitely generated hypergroup. If F is a symmetric generator of H such that $|F^{*n}| \leq Dn^d$ for some $d, D > 0$ and $\sigma_{\alpha,C}$ is an exponential weight on H for some $0 < \alpha < 1$ and $C > 0$. Then $\ell^1(H, \sigma_{\alpha,C})$ is injective and equivalently isomorphic to an operator algebra.

Proof. Let ω_β be the weight defined in Lemma 5.10. We define a function $\omega : H \rightarrow (0, \infty)$ by

$$\omega(x) := \frac{\sigma_{\alpha,C}(x)}{\omega_\beta(x)} = e^{C\tau_F(x)^\alpha - \beta \ln(1+\tau_F(x))} \quad (x \in H)$$

where ω_β is the polynomial weight defined on H associated to F and

$$\beta > \max\left\{1, \frac{6}{C\alpha(1-\alpha)}, \frac{d+1}{2}\right\}.$$

Therefore, by Lemma 5.10, $\omega(t) \leq M\omega(x)\omega(y)$ for some $M > 0$ and all $t, x, y \in H$ such that $t \in x * y$. Therefore

$$\frac{\sigma_{\alpha,C}(t)}{\sigma_{\alpha,C}(x)\sigma_{\alpha,C}(y)} \leq M \frac{\omega_\beta(t)}{\omega_\beta(x)\omega_\beta(y)}.$$

Therefore,

$$\frac{\sigma_{\alpha,C}(t)}{\sigma_{\alpha,C}(x)\sigma_{\alpha,C}(y)} \leq M' \left(\frac{1}{(1+\tau(x))^\beta} + \frac{1}{(1+\tau(y))^\beta} \right)$$

for a modified constant $M' > 0$. Therefore by the proof of Corollary 5.8, $\Omega_{\sigma_{\alpha,C}} \in T^2(H)$. Now Theorem 5.2 finishes the proof. \square

Example 5.12 As a result of Theorem 5.11, and to follow Example 5.9, if H is a polynomial hypergroup on \mathbb{N}_0 , for each exponential weight $\sigma_{\alpha,C}$ for $0 < \alpha < 1$ and $C > 0$, $\ell^1(H, \sigma_{\alpha,C})$ is injective. Note that this class of hypergroups includes $\widehat{\text{SU}}(2)$.

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A Appendix: Lifting weights from \mathbb{Z} to weights on $A(\mathrm{SU}(2))$

In this appendix, we briefly present a method to construct non-central weights on $A(\mathrm{SU}(2))$ which are related to Example 3.16. Here λ_G denotes the left regular representation of a compact group G on $L^2(G)$ and $VN(G)$ denotes the group von Neumann algebra generated by λ_G . We also identify \mathbb{T} with the (closed) subgroup of all matrices

$$\begin{bmatrix} t & 0 \\ 0 & \bar{t} \end{bmatrix}, \quad (t \in \mathbb{T})$$

in $\mathrm{SU}(2)$. It is an immediate consequence of Herz's restriction theorem that there is a canonical embedding of $VN(\mathbb{T})$ into $VN(\mathrm{SU}(2))$. More precisely, the mapping $\Gamma : VN(\mathbb{T}) \rightarrow VN(\mathrm{SU}(2))$ defined by

$$\Gamma(\lambda_{\mathbb{T}}(f)) = \int_{\mathbb{T}} f(t) \lambda_{\mathrm{SU}(2)}(f) dt$$

is a weak*-weak* isometric *-algebra homomorphism. We note that the integration in the definition of Γ is the Bochner integration in the weak operator topology of $B(L^2(\mathrm{SU}(2)))$.

Now suppose that σ is a (group) weight on \mathbb{Z} which is bounded below by some $\delta > 0$, i.e. σ^{-1} belongs to $\ell^\infty(\mathbb{Z})$. Through the Fourier transform \mathcal{F} on \mathbb{Z} , $\mathcal{F}(\sigma^{-1})$ is an element in $VN(\mathbb{T})$ defined by

$$\mathcal{F}(\sigma^{-1})\chi_k = \sigma(k)^{-1}\chi_k,$$

where $\chi_k(t) = t^k$ ($k \in \mathbb{Z}$) are the characters of \mathbb{T} .

We now consider the element $\Gamma(\mathcal{F}(\sigma^{-1}))$ in $VN(\mathrm{SU}(2))$. Since $\mathrm{SU}(2)$ is compact, we can write $\lambda_{\mathrm{SU}(2)}$ as the direct sum of the irreducible unitary representations of $\mathrm{SU}(2)$. Moreover, if we take $\widehat{\mathrm{SU}}(2) = \{\pi_\ell : \ell \in \mathbb{N}_0\}$, where each π_ℓ is a representation of dimension $\ell + 1$, then, by a straightforward computation based on [14, Theorem 29.18], we have that $\Gamma(\mathcal{F}(\sigma^{-1}))(\pi_\ell)$ is the

diagonal matrix $\text{diag}(\sigma(-\ell)^{-1}, \sigma(-\ell+2)^{-1}, \dots, \sigma(\ell-2)^{-1}, \sigma(\ell)^{-1})$. Therefore, if we define,

$$W = \bigoplus_{\ell} - \begin{bmatrix} \sigma(-\ell) & 0 & \cdots & 0 & 0 \\ 0 & \sigma(-\ell+2) & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \sigma(\ell-2) & 0 \\ 0 & 0 & \cdots & 0 & \sigma(\ell) \end{bmatrix}, \quad (\text{A.1})$$

then W is a (possibly unbounded) operator $L^2(\text{SU}(2))$ with $W^{-1} = \Gamma(\mathcal{F}(\sigma^{-1})) \in VN(\text{SU}(2))$. Moreover, it is straightforward to check that W satisfies the assumptions in [18, Definition 2.4] so that, in particular, it is a Fourier algebra weight on $A(\text{SU}(2))$. We can apply the formula in Definition 3.13 to W and define the (hypergroup) weight ω_{σ} on $\text{SU}(2)$ presented in Example 3.16.

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Mahmood Alaghmandan

Department of Mathematical Sciences, Chalmers University of Technology and University of Gothenburg, Gothenburg SE-412 96, Sweden
 mahala@chalmers.se

Ebrahim Samei

Department of Mathematics and Statistics, University of Saskatchewan, 142 Wiggins road, Saskatoon, SK S7N 5E6, Canada
 samei@math.usask.ca